On representations of the rotation group and magnetic monopoles

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Recently (Phys. Lett. **A302** (2002) 253, hep-th/0208210; hep-th/0403146) employing bounded infinite-dimensional representations of the rotation group we have argued that one can obtain the consistent monopole theory with generalized Dirac quantization condition, $2\kappa\mu \in \mathbb{Z}$, where κ is the weight of the Dirac string. Here we extend this proof to the unbounded infinite-dimensional representations.

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I. INTRODUCTION

The Dirac quantization relation [1] between an electric charge e and magnetic charge q,

$$2\mu = n,\ n \in \mathbb{Z}, \tag{1}$$

where $\mu = eq$, and we set $\hbar = c = 1$, has been obtained from various approaches based on quantum mechanics and quantum field theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. One of the widely accepted proofs of the Dirac selection rule is based on group representation theory and consists in the following: In the presence of magnetic monopole the operator of the total angular momentum

$$\mathbf{J} = \mathbf{r} \times (-i\nabla - e\mathbf{A}) - \mu\frac{\mathbf{r}}{r}, \tag{2}$$

has the same properties as a standard angular momentum and for any value of μ obeys the usual commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{3}$$

The requirement that J_i generate a finite-dimensional representation of the rotation group yields 2μ being integer and only values $2\mu = 0, \pm1, \pm2, \dots$ are allowed (for details see, for example, [3, 5, 7, 8, 9]).

Actually the charge quantization does not follow from the quantum-mechanical consideration and rotation invariance alone. Any treatment uses some additional assumptions that may be not physically inevitable.

Recently we have exploited this problem employing bounded infinite-dimensional representations of the rotation group and nonassociative gauge transformations. We argued that one can relax Dirac's condition and obtain the consistent monopole theory with the generalized quantization condition, $2\kappa\mu \in \mathbb{Z}$, κ being the weight of the Dirac string [12, 13]. In our Letter we extend this proof to the unbounded infinite-dimensional representations of the rotation group.

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II. MAGNETIC MONOPOLE PRELIMINARIES

As well-known any vector potential $\mathbf{A}$ being compatible with a magnetic field $\mathbf{B} = q\mathbf{r}/r^3$ of Dirac monopole must be singular on the string (the so-called *Dirac string*, further it will be denoted as $S_{\mathbf{n}}$), and one can write

$$\mathbf{B} = \mathrm{rot}\mathbf{A}_{\mathbf{n}} + \mathbf{h}_{\mathbf{n}}$$

where $\mathbf{h}_{\mathbf{n}}$ is the magnetic field of the Dirac string given by

$$\mathbf{h}_{\mathbf{n}} = 4\pi q\mathbf{n}\int_0^\infty \delta^3(\mathbf{r} - \mathbf{n}\tau)d\tau. \tag{4}$$

The unit vector $\mathbf{n}$ determines the direction of a string $S_{\mathbf{n}}$ passing from the origin of coordinates to ∞.

For instance, Dirac's original vector potential reads

$$\mathbf{A}_{\mathbf{n}} = q\frac{\mathbf{r} \times \mathbf{n}}{r(r - \mathbf{n}\cdot\mathbf{r})}, \tag{5}$$

and the Schwinger's choice is

$$\mathbf{A}^{SW} = \frac{1}{2}\big(\mathbf{A}_{\mathbf{n}} + \mathbf{A}_{-\mathbf{n}}\big), \tag{6}$$

the string being propagated from $-\infty$ to ∞ [6]. Both vector potentials yield the same magnetic monopole field, however the quantization is different. The Dirac condition is $2\mu = p$, while the Schwinger one is $\mu = p$, $p \in \mathbb{Z}$.

These two strings belong to a family $\{\mathbf{S}^\kappa_{\mathbf{n}}\}$ of *weighted strings*, κ being the weight of the semi-infinite Dirac string [12, 13]. The respective vector potential is defined as

$$\mathbf{A}^\kappa_{\mathbf{n}} = \kappa\mathbf{A}_{\mathbf{n}} + (1-\kappa)\mathbf{A}_{-\mathbf{n}}, \tag{7}$$

and the magnetic field of the string $\mathbf{S}^\kappa_{\mathbf{n}}$ is

$$\mathbf{h}^\kappa_{\mathbf{n}} = \kappa\mathbf{h}_{\mathbf{n}} + (1-\kappa)\mathbf{h}_{-\mathbf{n}} \tag{8}$$

Since $\mathbf{A}^\kappa_{-\mathbf{n}} = \mathbf{A}^{1-\kappa}_{\mathbf{n}}$, we obtain the following equivalence relation: $S^\kappa_{-\mathbf{n}} \simeq S^{1-\kappa}_{\mathbf{n}}$.

Two strings $S^\kappa_{\mathbf{n}}$ and $S^\kappa_{\mathbf{n}'}$ are related by the gauge transformation

$$A^{\kappa'}_{\mathbf{n}'} = A^\kappa_{\mathbf{n}} + d\chi. \tag{9}$$

and vice versa. An arbitrary transformation of the strings $S^\kappa_{\mathbf{n}} \to S^{\kappa'}_{\mathbf{n}'}$ can be realized as combination of $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}'}$ and $S^\kappa_{\mathbf{n}} \to S^{\kappa'}_{\mathbf{n}}$, where the first transformation is rotation, and the second one results in changing of the weight string $\kappa \to \kappa'$ without changing its orientation.

Let denote by $\mathbf{n}' = g\mathbf{n}, g \in \mathrm{SO}(3)$, the left action of the rotation group induced by $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}'}$. From rotational symmetry of the theory it follows this gauge transformation $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}'}$ can be undone by rotation $\mathbf{r} \to \mathbf{r}g$ as follows

$$A^\kappa_{\mathbf{n}'}(\mathbf{r}) = A^\kappa_{\mathbf{n}}(\mathbf{r}') = A^\kappa_{\mathbf{n}}(\mathbf{r}) + d\alpha((\mathbf{r}; g)), \tag{10}$$

$$\alpha(\mathbf{r}; g) = e\int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A}^\kappa_{\mathbf{n}}(\boldsymbol{\xi})\cdot d\boldsymbol{\xi}, \quad \mathbf{r}' = \mathbf{r}g \tag{11}$$

where the integration is performed along the geodesic $\widehat{\mathbf{r}\,\mathbf{r}'} \subset S^2$.

The transformation of the string $S^\kappa_{\mathbf{n}} \to S^{\kappa'}_{\mathbf{n}}$ is given by

$$A^{\kappa'}_{\mathbf{n}} = A^\kappa_{\mathbf{n}} - d\chi_{\mathbf{n}}, \tag{12}$$

$$d\chi_{\mathbf{n}} = 2q(\kappa' - \kappa)\frac{(\mathbf{r}\times\mathbf{n})\cdot d\mathbf{r}}{r^2 - (\mathbf{n}\cdot\mathbf{r})^2}, \tag{13}$$

where $\chi_{\mathbf{n}}$ is polar angle in the plane orthogonal to $\mathbf{n}$. This type of gauge transformations being singular one can be undone by combination of the inversion $\mathbf{r} \to -\mathbf{r}$ and $\mu \to -\mu$. In particular, if $\kappa' = 1 - \kappa$ we obtain the mirror string: $S^\kappa_{\mathbf{n}} \to S^\kappa_{-\mathbf{n}} \simeq S^{1-\kappa}_{\mathbf{n}}$.

III. REPRESENTATIONS OF THE ROTATION GROUP AND DIRAC'S QUANTIZATION CONDITION

Let ψ^ℓ_ν be an eigenvector of the operators J_3 and J^2:

$$J_3\psi^\lambda_\nu = \nu\psi^\ell_\nu, \quad J^2\psi^\ell_\nu = \ell(\ell+1)\psi^\ell_\nu, \tag{14}$$

ν, ℓ being real numbers. Involving the operators $J_\pm = J_1 \pm J_2$ it is easy to show that the spectrum of J_3 has the form $\nu = \nu_0 + n$, where $n = 0, \pm1, \pm2, \dots$.

Each irreducible representation is characterized by an eigenvalue of Casimir operator and the spectrum of the operator J_3. There are four distinct classes of representations [17, 18, 19]:

- *Representations unbounded from above and below*, in this case neither $\ell + \nu_0$ nor $\ell - \nu_0$ can be integers.
- *Representations bounded below*, with $\ell + \nu_0$ being an integer, and $\ell - \nu_0$ not equal to an integer.
- *Representations bounded above,* with $\ell - \nu_0$ being an integer, and $\ell + \nu_0$ not equal to an integer.
- *Representations bounded from above and below,* with $\ell - \nu_0$ and $\ell + \nu_0$ both being integers, that yields $\ell = k/2, \quad k \in \mathbb{Z}_+$.

The nonequivalent representations in each of the series of irreducible representations are denoted respectively by $D(\ell, \nu_0)$, $D^+(\ell)$, $D^-(\ell)$ and $D(k/2)$. The representations $D(\ell, \nu_0)$, $D^+(\ell)$ and $D^-(\ell)$ are infinite dimensional; $D(k/2)$ is (k+1)-dimensional representation. The representations $D^\pm(\ell)$ and $D(\ell, \nu_0)$ are discussed in detail in [14, 15, 16, 17, 18].

In fact the representations $D(\ell, \nu)$ and $D(-\ell-1, \nu)$, yielding the same value $Q = \ell(\ell+1)$ of the Casimir operator, are equivalent and the inequivalent representations may be labeled as $D(Q, \nu)$ [19]. If there exists the number $p_0 \in \mathbb{Z}$ such that $\nu + p_0 = \ell$, we have $J_+|\ell,\ell\rangle = 0$ and the representation becomes bounded above. In the similar manner if for a number $p_1 \in \mathbb{Z}$ one has $\nu + p_1 = -\ell$, then $J_-|\ell,-\ell\rangle = 0$ and the representation reduces to the bounded below. Finally, finite-dimensional unitary representation arises when there exist possibility of finding $J_+|\ell,\ell\rangle = 0$ and $J_-|\ell,-\ell\rangle = 0$. It is easy to see that in this case 2ℓ, $2m$ and 2ν all must be integers.

In what follows we will discuss the Dirac monopole problem within the framework of the representation theory outlined above.

Taking into account the spherical symmetry of the system, the vector potential can be taken as [10, 11]

$$A_N = q(1-\cos\theta)d\varphi, \quad A_S = -q(1+\cos\theta)d\varphi \tag{15}$$

where (r, θ, φ) are the spherical coordinates, and while A_N has singularity on the south pole of the sphere, A_S on the north one. In the overlap of the neghborhoods covering the sphere S^2 the potentials A_N and A_S are related by the following gauge transformation:

$$A_S = A_N - 2qd\varphi.$$

This is the particular case of transformation given by Eq. (12), when $\kappa = 0$ and $\kappa' = 1$.

We start by choosing the vector potential as

$$A = q(1-\cos\theta)d\varphi.$$

Then for the operators J_i's we have

$$J_\pm = e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi} - \frac{\mu\sin\theta}{1+\cos\theta}\right), \tag{16}$$

$$J_3 = -i\frac{\partial}{\partial\varphi} - \mu, \tag{17}$$

$$\mathbf{J}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \\ + \frac{2i\mu}{1+\cos\theta}\frac{\partial}{\partial\varphi} + \mu^2\frac{1-\cos\theta}{1+\cos\theta} + \mu^2. \tag{18}$$

Substituting the wave function $\Psi = R(r)Y(\theta,\varphi)$ into Schrödinger's equation

$$\hat{H}\Psi = E\Psi, \tag{19}$$

we obtain for the angular part the following equation:

$$\mathbf{J}^2 Y(\theta,\varphi) = \ell(\ell+1) Y(\theta,\varphi). \tag{20}$$

Starting with $J_3 Y = mY$ and assuming

$$Y = e^{i(m+\mu)\varphi} z^{(m+\mu)/2}(1-z)^{(m-\mu)/2} F(z), \tag{21}$$

where $z = (1-\cos\theta)/2$, we obtain the resultant equation in the standard form of the hypergeometric equation,

$$z(1-z)\frac{d^2F}{dz^2} + \big(c-(a+b+1)z\big)\frac{dF}{dz} - abF = 0, \tag{22}$$
$$a = m-\ell,\ b = m+\ell+1,\ c = m+\mu+1.$$

The hypergeometric function $F(a,b;c;z)$ reduces to a polynomial of degree n in z when a or b is equal to $-n$, $(n = 0,1,2,\dots)$ [**?** **?**], and the respective solution of Eq.(20) is of the form

$$Y_n^{(\delta,\gamma)}(u) = C_n\,(1-u)^{\delta/2}(1+u)^{\gamma/2} P_n^{(\delta,\gamma)}(u), \tag{23}$$

$P_n^{(\delta,\gamma)}(u)$ being the Jacobi polynomials, $u = \cos\theta$, and the normalization constant C is given by

$$C_n = \left(\left|\frac{2\pi\, 2^{\delta+\gamma+1}\Gamma(n+\delta+1)\Gamma(n+\gamma+1)}{\Gamma(n+1)\Gamma(n+\delta+\gamma+1)}\right|\right)^{-1/2}$$

The functions $Y_n^{(\delta,\gamma)}(u)$ form the basis of the representation bounded above or below. This case has been studied in detail in [12, 13].

If both of a and b are negative integers, that is $m+\ell = -p,\ m+\ell = -k,\ p,k \in \mathbb{Z}_+$, then the representation becomes finite dimensional. It is easy to check that in this case $m+\mu$ and $m-\mu$ must be integers, that yields the Dirac quantization condition $2\mu \in \mathbb{Z}$.

In the rest of the paper we will discuss the representation $D(\ell,\mu)$ unbounded above and below. We are looking for the solutions of the Eq. (20) such that being regular at the point $z = 0$, in general, can have singularity at $z = 1$, where the Dirac string crosses the sphere. As a result we obtain the following restrictions on the spectrum of the operator J_3:

$$m+\mu = n,\ n = 0,\pm1,\pm2,\dots. \tag{24}$$

The respective solution is given by

$$Y_\ell^{(\mu,n)} = C(\ell,\mu,n) e^{in\varphi} z^{n/2}(1-z)^{n/2-\mu} F(a,b,c;z), \tag{25}$$
$$a = n-\mu-\ell,\ b = n-\mu+\ell+1,\ c = 1+n,$$

where $C(\ell,\mu,n)$ is a suitable normalization constant (for the details of the normalization procedure see [13, 17, 18]).

Consider now the other choice of the vector potential

$$A = -q(1+\cos\theta)d\varphi,$$

which corresponds to the Dirac string crossing the sphere at north pole $(z = 0)$. In this case the solution $\tilde{Y}_\ell^{(\mu,n)}$ of the equation (20) being regular at the point $z = 1$ takes the same form as in Eq. (25)

$$\tilde{Y}_\ell^{(\mu,n)} = C(\ell,\mu,n) e^{in\varphi} z^{n/2+\mu}(1-z)^{n/2} F(a,b,c;1-z), \tag{26}$$
$$a = n+\mu-\ell,\ b = n+\mu+\ell+1,\ c = 1+n.$$

The spectrum of the operator J_3 being diferent from (24) is found to be

$$m-\mu = n,\ n = 0,\pm1,\pm2,\dots. \tag{27}$$

Notice that the functions $\tilde{Y}_\ell^{(\mu,n)}$ can be obtained from $Y_\ell^{(\mu,n)}$ by the change of $z \mapsto (1-z)$ and $\mu \mapsto -\mu$, that is agree with the gauge transformation

$$A_S = A_N - 2q d\varphi$$

(see also Eqs.(12),(13)) .

The set of the functions $\left\{\tilde{Y}_\ell^{(\mu,n)}, Y_\ell^{(\mu,n)}\right\}$ form the complete bi-orthonormal canonical basis of the representation $D(\ell,\mu)$ in the indefinite-metric Hilbert space with the indefinite metric given by [20]

$$\eta_{mm'} = (-1)^{\sigma(m)}\delta_{mm'}, \tag{28}$$

where

$$(-1)^{\sigma(m)} = \operatorname{sgn}\big(\Gamma(\ell-m+1)\Gamma(\ell+m+1)\big),$$

$\operatorname{sgn}(x)$ being the signum function. One can see that the spectrum of the operator J_3 is unbounded, double-degenerate and discrete.

The general case of an arbitrary weighted string $S^\kappa_{\mathbf{n}}$ can be considered in the following way: For $m \pm \mu = n$ the weighted solutions of the Schrödinger equation are given by

$$Y_{\kappa,\ell}^{(\mu,n)} = \mathrm{e}^{-2i\kappa\mu\varphi} Y_\ell^{(\mu,n)}, \quad m = n-\mu \tag{29}$$
$$\tilde{Y}_{\kappa,\ell}^{(\mu,n)} = \mathrm{e}^{-2i\kappa\mu\varphi} \tilde{Y}_\ell^{(\mu,n)}, \quad m = n+\mu. \tag{30}$$

Since a Dirac string may be rotated by gauge transformation the widely accepted point of view is that the string is unobservable. Thus, to avoid the appearence of an Aharonov-Bohm effect produced by a Dirac string, one has to impose the generalized Dirac quantiztion condition $2\kappa\mu \in \mathbb{Z}$. In particular cases $\kappa = 1$ and $\kappa = 1/2$ it yields the Dirac and Schwinger selectional rules respectively.

IV. CONCLUDING REMARKS

We have argued, by applying infinite-dimensional representations of the rotation group, that the Dirac quantization condition can be relaxed and changed by $2\kappa\mu \in \mathbb{Z}$, where κ is the weight of the Dirac string. This selectional

rule arises as natural condition of being consistent with an algebra of observables and ensures the absence of an Aharonov-Bohm effect produced by Dirac string. Moreover, since there is no any restriction on the parameter κ, an arbitrary magnetic charge is allowed.

It follows from our description that the spectrum of the operator J_3 is double-degenerate, discrete and unbounded, $m = n \pm \mu$. The physical interpretation of this result is not clear yet. We believe that it can be explained treating the charge-monopole system as a free anyon with translational and spin degrees of freedom [21].

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